Worksheet for 2021-09-22

Conceptual questions

Question 1. Suppose $f(x, y)$ is a function of two variables with $\nabla f(2, 5) = \langle 4, 0\rangle$. How many unit vectors $\mathbf{u}$ are there such that $D_{\mathbf{u}} f(2, 5)$ is -4? What about -3? 0? 5?

Question 2. Suppose $\nabla f(a, b) = 0$. In the second derivative test as written on page 961 in the textbook, they say "if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum."

Is it significant that we look at f_{xx} rather than f_{yy}?

Computations

Problem 1. Consider a point $P = (a, b, c)$ on the surface S defined by $xyz = 6$. The tangent plane H at this point meets the coordinate axes at the points $(A, 0, 0)$, $(0, B, 0)$, $(0, 0, C)$. In other words, A, B, C are the x, y, z-intercepts of H, respectively.

(a) Compute each of A, B, C in terms of a, b, c only.
(b) Show that their product ABC is independent of a, b, c. What is it equal to?

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1. To take a "coordinate" approach, let $\mathbf{u} = \langle a, b\rangle$. Then we could write

$$D_{\mathbf{u}}(2,5) = \langle 4, 0\rangle \cdot \langle a, b\rangle = 4a$$

and we also have the equation $a^2 + b^2 = 1^2 = 1$ because $\mathbf{u}$ is a unit vector. The system of equations

$$4a = -4$$
$$a^2 + b^2 = 1$$

has a single solution: $a = -1, b = 0$. Replacing the -4 by $-3, 0, 5$ gives $2, 2, 0$ solutions, respectively.

Alternatively, we could write

$$D_{\mathbf{u}}(2,5) = |\nabla f(2,5)||\mathbf{u}|\cos\theta = 4\cos\theta$$

where θ is the angle between $\nabla f(2,5)$ and $\mathbf{u}$. Keep in mind, however, that there is *one* unit vector that would make θ equal to 0 or π, whereas there are *two* unit vectors that would make θ equal to any value strictly inbetween (so a single solution for θ may correspond to multiple solutions for $\mathbf{u}$).

Question 2. We are assuming that $D(a,b) > 0$, which is to say

$$f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2 > 0.$$

Rearranging this gives

$$f_{xx}(a,b)f_{yy}(a,b) > f_{xy}(a,b)^2 \geq 0$$

since a square is always nonnegative. Hence the product on the left is *strictly* positive, and both terms must have the same sign. So requiring that $f_{xx}(a,b) > 0$ is equivalent to requiring that $f_{yy}(a,b) > 0$.

Note that we are **not** saying that f_{xx} and f_{yy} are equal; just that they must have the same sign in the case $D > 0$.

Answers to computations

Problem 1.

(a) Let $F(x,y,z) = xyz$, so that S is the 6 level set of F. We have $\nabla F(a,b,c) = \langle bc, ac, ab\rangle$, and thus an equation of H is

$$bc(x-a) + ac(y-b) + ab(z-c) = 0$$

which simplifies to $bcx + acy + abz = 18$. (Remember, $abc = 6$ because (a,b,c) lies on the surface S! Also, a, b, c are constants, so this is indeed a plane equation.)

As $(A, 0, 0)$ is a point on the plane, we must have $bcA = 18$, and thus $A = 18/bc$. Similar computations show $B = 18/ac$ and $C = 18/ab$.

(b) The product is

$$ABC = \frac{18^3}{a^2b^2c^2} = \frac{18^3}{6^2} = 162.$$

This is independent of the choice of a, b, c, as claimed.

This problem (with arbitrary k in place of 6) was Problem 10 on the Fall 2020 Midterm 1.